Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 11: Random Processes

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Random Processes

1) **The Random Process Concept**
2) **Stationary and Independence**
3) **Correlation Functions**

1) The Random Process Concept

- ✓ The concept of a **random process** based on **enlarging** the random variable concept to include **time**.
- ✓ Let **s** denote the random outcome of an experiment. To every such outcome suppose a waveform **x (t, s)** is assigned.
- ✓ The **family** of all such **waveforms** form a random process **X (t, s).**
- ✓ For fixed outcome s_i , **X(t , s)** is a specific time function.
  - For fixed **t,**

$$X_1 = X(t_1, s_i)$$

is a random variable. The **ensemble** of all such **realizations** **X (t, s)** over time represents the **random (stochastic) process** **X(t).**

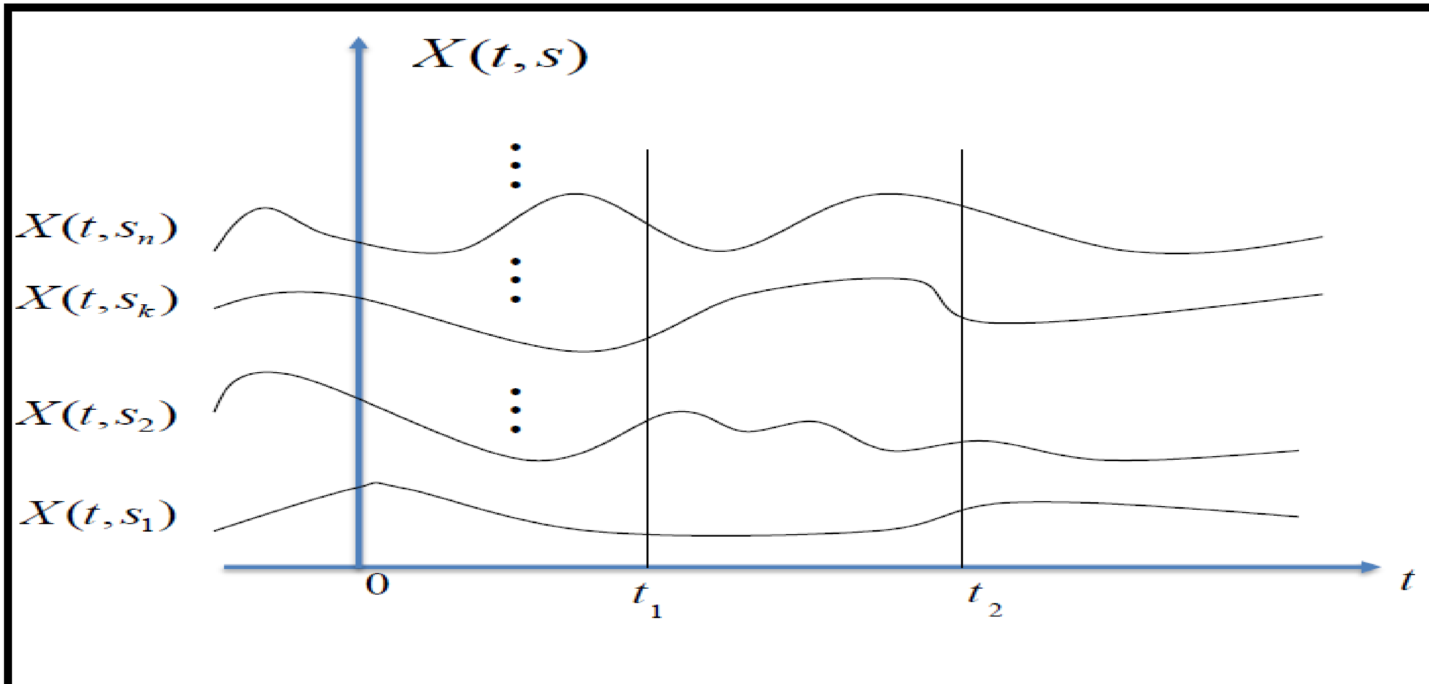

Fig. A continuous random process

Classification of Random Processes:

- ✓ We shall consider four cases of random processes based on **t** and **X**

  1) Continuous Random Processes:

  If **X** and **t** are continuous. Example in **Fig.6-1**

  2) Discrete Random Processes:

  If **X** has only discrete values and **t** is continuous. Example in **Fig.6-2**

3) Continuous Random Sequence:

If **X** is continuous and **t** has only discrete values. Example in **Fig.6-3**

4) Discrete Random Sequence:

If **X** and **t** are both, have only discrete values. Example in **Fig.6-4**

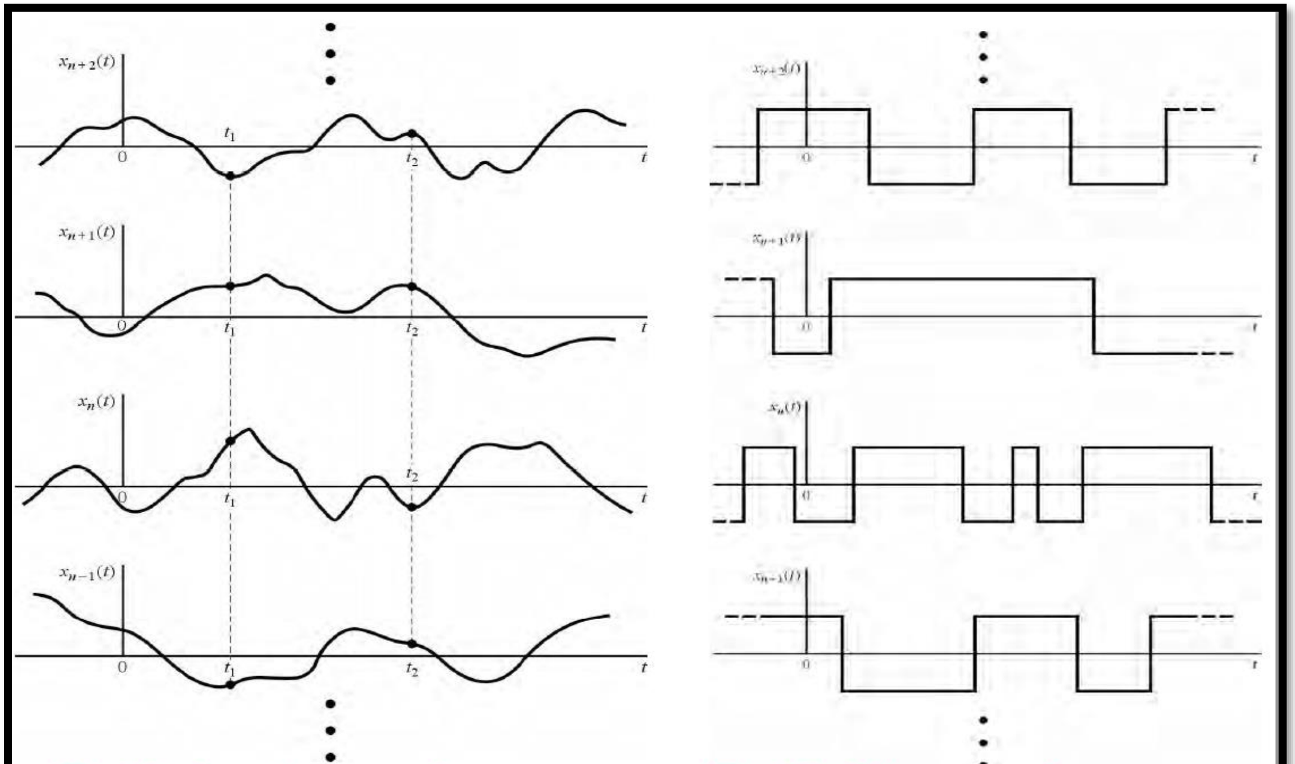

Fig.6-1 A continuous random process   Fig.6-2 A discrete random process

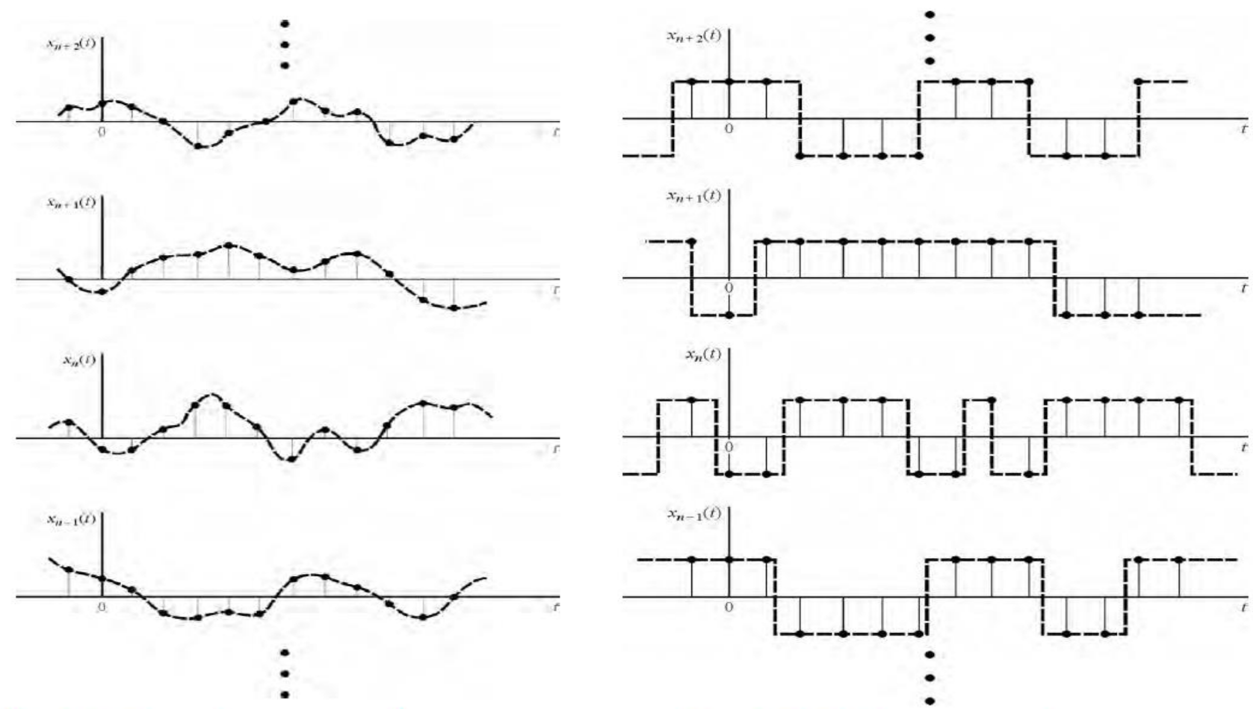

Fig.6-3 A continuous random sequence Fig.6-4 A discrete random sequence

2) Stationary and Independence

- ✓A random process is said to be **stationary** if all its **statistical properties do not change with time**

Distribution and Density Functions:

- ✓If **X (t)** is a **stochastic** process, then for fixed **t, X (t)** represents a random variable. Its distribution function (**CDF**) given by

$$F_x(x_1;t_1) = P\{X(t_1) \le x_1\}$$

- ✓ For **two random** variables, the **second-order joint distribution** function is

$$F_x(x_1,x_2;t_1,t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

- ✓ For **N random** variables, the **N_{th}-order** joint distribution function is

$$F_x(x_1,\ldots,x_N;t_1,\ldots,t_N) = P\{X(t_1) \le x_1,\ldots, X(t_N) \le x_N\}$$

- ✓ **Joint density functions** of interest are

$$f_x(x_1;t_1) = \frac{dF_x(x_1;t_1)}{dx_1}$$

$$f_x(x_1,x_2;t_1,t_2) = \frac{\partial^2 F_x(x_1,x_2;t_1,t_2)}{\partial x_1 \partial x_2}$$

$$f_x(x_1,\ldots,x_N;t_1,\ldots,t_N) = \frac{\partial^N F_x(x_1,\ldots,x_N;t_1,\ldots,t_N)}{\partial x_1 \ldots \partial x_N}$$

First Order Stationary Process:

- ✓ A random process is called **stationary to order one** if its first-order density function does not change with a shift in time origin

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$$

- ✓ A consequence of the above condition is that the process mean value is a constant:

$$E[X(t)] = \overline{X} = \text{constant}$$

Second Order Stationary Process:

- ✓ A random process is called **stationary to order two** if its second-order density function is a function of time difference and not the absolute time.

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;t_1+\Delta,t_2+\Delta)$$

- ✓ A **second-order stationary process is also first-order stationary** because the second-order density function determines the lower, first-order, density
- ✓ A consequence of the above condition is that the **autocorrelation function of a second-order stationary process is a function only of time difference and not absolute time**:

$$R_{XX}(t_1, t_2) = R_{XX}(t_1, t_1 + \tau) = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)$$

Wide-Sense Stationary Process:

- ✓ A random process is called **wide-sense stationary (WSS)** if the two following conditions are true

$$E[X(t)] = \overline{X} = \text{constant}$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

- ✓ **Example:** Show that the random process

$$X(t) = A\cos(\omega_0 t + \Theta)$$

is wide-sense stationary if $\mathbf{A}$ and $\boldsymbol{\omega_0}$ are constants and $\boldsymbol{\Theta}$ is a uniformly distributed random variable on the interval $(0, 2\pi)$.

- **Solution:**

The mean value

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \Theta)\frac{1}{2\pi}\,d\Theta = 0$$

The autocorrelation function

$$\begin{aligned}
R_{xx}(t, t+\tau) &= E[A\cos(\omega_0 t + \Theta)A\cos(\omega_0 t + \omega_0\tau + \Theta)] \\
&= \frac{A^2}{2}E[\cos\omega_0\tau + \cos(2\omega_0 t + \omega_0\tau + 2\Theta)] \\
&= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\Theta)] \\
&= \frac{A^2}{2}\cos\omega_0\tau
\end{aligned}$$

Since

$$E[X(t)] = \text{constant}$$

And

$$R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

Then the random process is wide-sense stationary

N-Order and Strict-Sense Stationary:

- ✓ A random process is called **stationary to order N** if its **N_{th}-order density function** does not change with a shift in time origin; that is

$$f_x(x_1,\ldots,x_N;t_1,\ldots,t_N)=f_x(x_1,\ldots,x_N;t_1+\Delta,\ldots,t_N+\Delta)$$

$$\text{for all } t_1,t_2,\ldots,t_N \text{ and } \Delta$$

- ✓ A process stationary to all orders **N=1, 2,...,** is called **strict-sense stationary**.

Time Average and Ergodicity:

- ✓ The **time average** of a quantity is defined as

$$A[\,\cdot\,]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\,\cdot\,]dt$$

- ✓ **A** is used to denote **time average** in a manner analogous to **E** for the **statistical average**.
- ✓ The **mean value** and the **autocorrelation function** of a sample function **x(t)** are defined by

$$\overline{x} = A[\,x(t)\,] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\Re_{XX}(\tau) = A[\,x(t)x(t+\tau)\,] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Note: A lower case letter is used to imply a sample function

- ✓ For any **one sample function** of the random process **X(t)**, if the following two conditions are satisfied then the random process is called **ergodic process**.

$$E[\overline{x}] = \overline{x} = \overline{X}$$
$$E[\Re_{XX}(\tau)] = \Re_{XX}(\tau) = R_{XX}(\tau)$$

- ✓ **Ergodicity** is a **very restrictive form of stationary**, and it may be difficult to prove that it constitutes a reasonable assumption in any physical situation.
- ✓ Nevertheless, we shall **often assume a process is ergodic to simplify problems**.

3) Correlation Functions

Autocorrelation Function and Its Properties:

✓ The **autocorrelation** of a random process **X(t)** is given by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

Assume that : $t_1 = t$ and $t_2 = t + \tau$

✓ If **X(t)** is at least wide-sense stationary then

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Autocorrelation Function Properties:

1. $|R_{XX}(\tau)| \le R_{XX}(0)$
2. $R_{XX}(-\tau) = R_{XX}(\tau)$
3. $R_{XX}(0) = E[X^2(t)]$
4. If $E[X(t)] = \overline{X} \neq 0$ and $X(t)$ is ergodic with no periodic components, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$
5. If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
6. If $X(t)$ is ergodic, zero mean, and has no periodic components, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$
7. $R_{XX}(\tau)$ cannot have an arbitrary shape.

✓ **Example:** Given the **autocorrelation** function, for a **stationary ergodic** process with **no periodic** components, is

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Find the **mean value** and **variance** of the process **X(t)**.

- **Solution:**

The mean value

$$\overline{X}^2 = \lim_{|\tau|\to\infty} R_{xx}(\tau) = 25 \quad \text{then} \quad \overline{X} = \pm 5$$

The mean square value

$$E[X^2(t)] = R_{xx}(0) = 25 + 4 = 29$$

And the variance

$$\sigma_X^2 = E[X^2(t)] - \overline{X}^2 = 29 - 25 = 4$$

Cross-Correlation Function and Its Properties:

- ✓ The **cross-correlation** of two random processes $X(t)$ and $Y(t)$ is given by

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

Assume that :
$t_1 = t$ and $t_2 = t+\tau$

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

- ✓ If $R_{XY}(\tau) = 0$ then $X(t)$ and $Y(t)$ are called **orthogonal processes**
- ✓ If $X(t)$ and $Y(t)$ are at **least jointly wide-sense stationary** then

$$R_{XY}(\tau) = E[X(t)]E[Y(t+\tau)] = \overline{X}\,\overline{Y}$$

Cross-Correlation Function Properties:

1. $R_{XY}(-\tau) = R_{YX}(\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)}$
3. $|R_{XY}(\tau)| \le \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$

✓ **Example:** Let two random processes $X(t)$ and $Y(t)$ be defined by

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

Where A and B are random variables and ω_0 is a constant.
If A and B are **uncorrelated**, **zero-mean** random variables with the **same variance**. Find the **cross-correlation** of $X(t)$ and $Y(t)$.

- **Solution:**

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$
$$= E[\{A\cos(\omega_0 t) + B\sin(\omega_0 t)\}\{B\cos(\omega_0 t + \omega_0\tau) - A\sin(\omega_0 t + \omega_0\tau)\}]$$
$$= E[AB\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) + B^2\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)$$
$$- A^2\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau) - AB\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)]$$
$$= E[AB]\cos(2\omega_0 t + \omega_0\tau) + E[B^2]\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)$$
$$- E[A^2]\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$$

but

$$E[A] = 0, \quad E[B] = 0, \quad E[AB] = 0, \text{ and } E[A^2] = E[B^2] = \sigma^2$$

we get

$$R_{XY}(t, t+\tau) = -\sigma^2\sin(\omega_0\tau)$$

Since $R_{XY}(t, t+\tau) = R_{XY}(\tau)$ then X(t) and Y(t) are jointly WSS.

Covariance Functions:

- ✓The **auto-covariance function** of two random processes $X(t)$ and $Y(t)$ is given by

$$C_{XX}(t,t+\tau) = E[\{X(t)-E[X(t)]\}\{X(t+\tau)-E[X(t+\tau)]\}]$$
$$= R_{XX}(t,t+\tau) - E[X(t)]E[X(t+\tau)]$$

- ✓The **cross-covariance function** of two random processes $X(t)$ and $Y(t)$ is given by

$$C_{XY}(t,t+\tau) = E[\{X(t)-E[X(t)]\}\{Y(t+\tau)-E[Y(t+\tau)]\}]$$
$$= R_{XY}(t,t+\tau) - E[X(t)]E[Y(t+\tau)]$$

- ✓If $X(t)$ and $Y(t)$ are at **least jointly wide-sense stationary** then

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$
$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{X}\,\overline{Y}$$